Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection

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Cycle classification

Let $\sigma \in \mathfrak{S}_n$. We classify indices $i \in [n]$ according to cycle classification:

- cycle valley if $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peak if $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

The Foata–Zeilberger bijection

The Forward Bijection $\sigma \mapsto (\omega, \xi)$

(a) Description of Motzkin path ω

Define a path $\omega = (\omega_0, \ldots, \omega_n)$ where $\omega_0 = (0,0)$ and $\omega_n = (n,0)$ as follows:

- If i is a cycle valley, $\omega_i = \omega_i + (1,0)$ (a rise $\nearrow$)
- If i is a cycle peak, $\omega_i = \omega_i + (1,-1)$ (a fall $\searrow$)
- If i is a cycle double rise, a cycle double fall or a fixed point, $\omega_i = \omega_i + (1,0)$ (a level step $\rightarrow$)

ω is a Motzkin path: it uses steps $\nearrow$, $\searrow$ and $\rightarrow$ starts at $(0,0)$, ends at $(n,0)$, and never goes below the x-axis.

(b) Description of Labels ξ_i for $i \in [n]$

$$\xi_i = \begin{cases} \#\{j:\ j < i \text{ and } \sigma(j) > \sigma(i)\} & \text{if } \sigma(i) > i \quad \text{if } i \in \text{Cval} \cup \text{Cdrise} \\ \#\{j:\ j > i \text{ and } \sigma(j) < \sigma(i)\} & \text{if } \sigma(i) < i \quad \text{if } i \in \text{Cpeak} \cup \text{Cdfall} \\ 0 & \text{if } \sigma(i) = i \quad \text{if } i \in \text{Fix} \end{cases}$$

Foata and Zeilberger (1990) showed that $\sigma \mapsto (\omega, \xi)$ is a bijection from permutations to labelled Motzkin paths where the level steps come in three colours (corresponding to fixed points, cycle double rises and cycle double falls).

The Inverse Bijection

We now describe the map $(\omega, \xi) \mapsto \sigma$. First define the sets

$$\begin{aligned} F &= \{i \in \sigma : \sigma(i) > i\} = \text{Cdrise} \cup \text{Cval} \\ F' &= \{i \in \sigma : i > \sigma^{-1}(i)\} = \text{Cdrise} \cup \text{Cpeak} \\ G &= \{i \in \sigma : \sigma(i) < i\} = \text{Cdfall} \cup \text{Cpeak} \\ G' &= \{i \in \sigma : i < \sigma^{-1}(i)\} = \text{Cdfall} \cup \text{Cval} \\ H &= \{i \in \sigma : i = \sigma(i)\} = \text{Fix} \end{aligned}$$

(a) We can clearly read off the sets F, F', G, G', H from the Motzkin path ω.

(b) We construct $\sigma|_F : F \to F'$. Let $F = \{i_1 < \ldots < i_k\}$. Let $j_1 j_2 \cdots j_k$ be the permutation of the letters F', such that for any α, the number of letters larger than j_α to the left of j_α is ξ_{i_α}. Then we define $\sigma(i_\alpha) = j_\alpha$ (this is the left-to-right inversion table).
We similarly define $\sigma|_G : G \to G'$, except now we look at letters smaller to the right (this is the right-to-inversion table).

Question

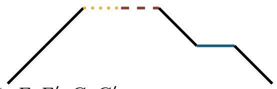
Can we keep track of the number of cycles in the Foata–Zeilberger bijection?

An Example

Let $\sigma = 5624173 = (15)(2673)(4) \in \mathfrak{S}_7$.

- Cval = $\{1, 2\}$
- Cpeak = $\{5, 7\}$
- Cdrise = $\{6\}$
- Cdfall = $\{3\}$
- Fix = $\{4\}$

The Motzkin path ω is



The labels ξ and the sets F, F', G, G' are:

$i \in F$	1	2	6
$\sigma(i) \in F'$	5	6	7
ξ_i	0	0	0

$i \in G$	3	5	7
$\sigma(i) \in G'$	2	1	3
ξ_i	1	0	0

Laguerre digraphs

A Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $[n]$ and has indegree 0 or 1 and outdegree 0 or 1. They were introduced by Foata–Strehl (1984), Sokal (2022) as a combinatorial interpretation of the coefficients of the Laguerre polynomials.
Connected components are either directed cycles or directed paths.
A Laguerre digraph without any path is a permutation in cycle notation.
They are sometimes also referred to as Laguerre configurations or partial permutations.

Foata–Zeilberger history and reinterpretation of inverse bijection

Start with Laguerre digraph with n vertices and no edges. At each step insert an edge $i \to \sigma(i)$ in the following stages:
Stage (a): Insert loops in increasing order of $i \in H =$ Fix.
At the end of this stage, the resulting Laguerre digraph consists of loops at all vertices in H. All other vertices have no adjacent edges.

Stage (b): Insert edges $i \in \sigma(i)$ in increasing order of $i \in G =$ Cdfall $\cup$ Cpeak.
At the end of this stage, the resulting Laguerre digraph consists of loops at all vertices in H, and directed paths with at least two vertices, in which the initial vertex of the path is a cycle peak, the final vertex is a cycle valley and the intermediate vertices (if any) are cycle double falls, and no edges adjacent to cycle double rises.

Stage (c): Insert edges $i \in \sigma(i)$ in *decreasing order* of $i \in F =$ Cdrise $\cup$ Cval.
We carefully study this stage and see that a cycle is closed when i is the final vertex of a path and $\sigma(i)$ is the initial vertex of the same path. We show that in such a situation, i is a valley.
This allows us to count cycles in the "history" of this bijection.

Applications a la Flajolet (1980)

Sokal–Zeng conjecture (2022)

An index i can also be classified using the **record classification**.

- record (or left-to-right maximum) if $\sigma(j) < \sigma(i)$ for all $j < i$;
- antirecord (or right-to-left minimum) if $\sigma(j) > \sigma(i)$ for all $j > i$;

Every index i is either an exclusive record, exclusive antirecord, a record-antirecord or a neither-record-antirecord.

Apply the record and cycle classifications simultaneously, to obtain 10 disjoint categories of the **record-and-cycle classification**: ereccval, ereccdrise, eareccpeak , eareccdfall, rar, nrcpeak, nrcval, nrcdrise, nrcdfall, nrfix.

Define the 11-variable polynomials

$$\widehat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w, \lambda) = \sum_{\sigma \in_n} x_1^{\text{eareccpeak}(\sigma)} x_2^{\text{eareccdfall}(\sigma)} y_1^{\text{ereccval}(\sigma)} y_2^{\text{ereccdrise}(\sigma)} z^{\text{rar}(\sigma)} \times u_1^{\text{nrcpeak}(\sigma)} u_2^{\text{nrcdfall}(\sigma)} v_1^{\text{nrcval}(\sigma)} v_2^{\text{nrcdrise}(\sigma)} w^{\text{nrfix}(\sigma)} \lambda^{\text{cyc}(\sigma)}$$

The following continued fraction was conjectured by Sokal–Zeng (2022):

Theorem 1

The ordinary generating function of the polynomials $\widehat{Q}_n$ specialised to $v_1 = y_1$ has the J-type continued fraction

$$\sum_{n=0}^{\infty} \widehat{Q}_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, \mathbf{w}, \lambda)\, t^n = \cfrac{1}{1 - \lambda w t - \cfrac{\lambda x_1 y_1 t^2}{1 - (x_2+y_2+\lambda w)t - \cfrac{(\lambda+1)(x_1+u_1)y_1 t^2}{1 - (x_2+y_2+u_2+v_2+\lambda w)t - \cfrac{(\lambda+2)(x_1+2u_1)y_1 t^2}{1 - \cdots}}}}$$

Randrianarivony–Zeng conjecture (1996)

Randrianarivony–Zeng (1996) introduced the following 4-variable polynomials counting four statistics comi, lema, cemi, remi

$$G_n(x, y, \bar{x}, \bar{y}) = \sum_{\sigma \in \mathfrak{D}^o_{2n}} x^{\text{comi}(\sigma)} y^{\text{lema}(\sigma)} \bar{x}^{\text{cemi}(\sigma)} \bar{y}^{\text{remi}(\sigma)}$$

Here the sum runs over all D-o-semiderangements: these are permutations of $2n$ where the indices satisfy $2k - 1 < \sigma(2k-1)$ and $2k \geq \sigma(2k)$. These are counted by the Genocchi numbers. Our interpretation also resolves the following:

Theorem 2

The ordinary generating function of the polynomials $G_n(x, y, \bar{x}, \bar{y})$ has the S-type continued fraction

$$1 + \sum_{n=1}^{\infty} G_n(x, y, \bar{x}, \bar{y}) t^n = \cfrac{1}{1 - \cfrac{xyt}{1 - \cfrac{1(\bar{x}+\bar{y})t}{1 - \cfrac{(x+1)(y+1)t}{1 - \cfrac{2(\bar{x}+\bar{y}+1)}{1 - \cfrac{(x+2)(y+2)t}{1 - \cfrac{3(\bar{x}+\bar{y}+2)t}{\cdots}}}}}}}$$

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