Coefficientwise Hankel total positivity: a survey

Bishal Deb
Yau Mathematical Sciences Center, Tsinghua University

June 22nd, 2026
Interactions between Combinatorics, Analysis and Statistical Physics
A conference in (belated) celebration of Alan Sokal's 70th birthday.

Collection of open problems and conjectures:
https://bishaldeb.com/positivity.html

1. Stieltjes moment sequences
2. Coefficientwise Hankel total positivity
3. Classical S and T fractions
4. Production matrix method
5. Exponential Riordan arrays
6. Multiple orthogonal polynomials
7. Toeplitz TP implying Hankel TP
8. An open problem
9. A new website

Hankel total positivity

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Hankel matrix:

$$(a_{i+j})_{i,j\geq 0} = \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\ a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\ a_3 & a_4 & a_5 & a_6 & a_7 & \ldots \\ a_4 & a_5 & a_6 & a_7 & a_8 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

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We say that a sequence $(a_n)_{n\geq 0}$ is Hankel-TP if its Hankel matrix is TP.

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$$a_n = \int_0^\infty x^n d\mu(x)$$

for all $n \geq 0$. (Stieltjes-moment sequence.)

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3. *There exists numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that*
$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}.$$

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4. Another statement involving counting walks on graphs (Flajolet(1980) + Elvey Price–Guttmann (2019)).

Many important combinatorial sequences are known to be Stieltjes moment sequences

- Catalan numbers
- $n!$.
- $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$
- Bell numbers
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A lot more are conjectured to be Stieltjes moment sequences

Permutation patterns conjecture

Let σ, π be two permutations. σ is said to *contain* pattern π if there is a subsequence of σ that has the same relative order as π.

Let $\mathrm{Av}_n(\pi)$ denote the set of all permutations in $\mathfrak{S}_n$ *avoiding* the pattern π.

Conjecture (Blitvić, Kammoun, Stiengrímsson, Bostan, Elvey Price, Guttmann, Maillard, Clisby, Conway, Inoue)

For any permutation π, the sequence $|\mathrm{Av}_n(\pi)|$ is Hankel TP.

Partially ordered commutative ring

Definition

A partially ordered commutative ring is a pair $(R, \mathcal{P})$ where R is a commutative ring with identity and $\mathcal{P}$ is a subset of R satisfying

(a) $0, 1 \in \mathcal{P}$

(b) If $a, b \in \mathcal{P}$, then $a + b \in \mathcal{P}$ and $ab \in \mathcal{P}$.

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Example: R is a polynomial ring containing $\mathbb{Q}$ and $\mathcal{P}$ are all polynomials with non-negative coefficients.

Definition

A matrix with entries in a partially ordered commutative ring R is called TP if all its minors belong to $\mathcal{P}$.

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Coefficientwise TP $\Longrightarrow$ Pointwise TP.
A sequence of polynomials is said to be coefficientwise Hankel TP if its Hankel matrix is coefficientwise TP.

Coefficientwise TP of Hankel matrix of a sequence $(p_n(x))_{n\geq 0}$ implies its coefficientwise log-convexity

Coefficientwise Hankel total positivity: a new subfield of study

Established as a new field of study in

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Before these, there were many papers on the study of coefficientwise log-convexity in combinatorics by the Dalian and Tianjin schools of combinatorics.

Stieltjes type continued fractions

Theorem (Sokal(2014), Pétréolle–Sokal–Zhu (2023))

Let $\boldsymbol{\alpha} = \alpha_1, \alpha_2, \ldots$ *be a sequence of indeterminates and let* $S_n(\boldsymbol{\alpha})$ *be a polynomial defined by*

$$\sum_{n=0}^{\infty} S_n(\boldsymbol{\alpha}) t^n := \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}.$$

Then $(S_n(\boldsymbol{\alpha}))_{n \geq 0}$ *is coefficientwise Hankel-TP.*

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Converse need not be true. Continued fraction only a sufficient condition to prove coefficientwise Hankel-TP.

Lattice path interpretation

Consider a Dyck path, let's say

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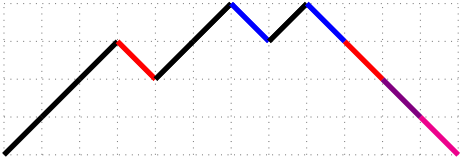

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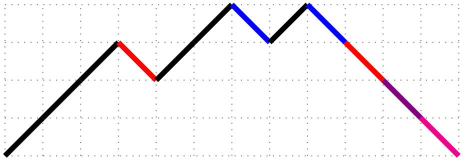

Assign weights:

- $\nearrow$ from height $(i-1) \to i$ — 1
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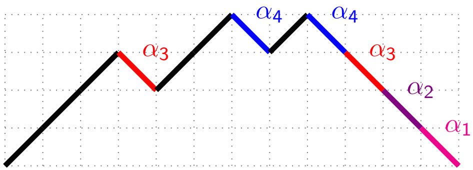

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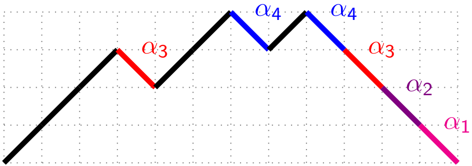

$$\text{Weight} = \alpha_1\alpha_2\alpha_3^2\alpha_4^2$$

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Theorem (Flajolet 1980)

Stieltjes-Rogers polynomial $S_n(\boldsymbol{\alpha})$ is the weighted sum over all Dyck paths of semilength n.

Guessing Hankel total positivity

Let $a_0, a_1, \ldots$ be a sequence of real numbers with generating function

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Given $a_0, a_1, \ldots, a_N$, we will compute $\alpha_0, \ldots, \alpha_N$.

If $\alpha_0, \ldots, \alpha_N > 0$, we *guess* that the sequence is Hankel TP.

Computing continued fraction

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But this recurrence is slow in practice due to the divisions of power series.

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Euler–Viscovatov algorithm: (Sokal, 2022)

Recursively define $B_j(t) = A_j(t)B_{j-1}(t)$ and $B_{-1}(t) = 1$. Then

$$\frac{B_j(t)}{B_{j-1}(t)} = \frac{\alpha_j}{1 - t\frac{B_{j+1}(t)}{B_j(t)}}$$

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We expand this and obtain

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Recursively compute $\alpha_j, B_{j+1}(t) + O(t^{N-j})$

$$\alpha_j \ = \ B_j(0)/B_{j-1}(0), \qquad B_{j+1}(t) \ = \ \frac{1}{t}(B_j(t) - \alpha_j B_{j-1}(t))$$

Slide from talk of Andrew Elvey Price, Permutation Patterns 2023

GUESSING STIELTJES-NESS WITH OEIS

We ran the Euler-Viskovatov algorithm on all 304698 OEIS sequences with at least 15 terms (only considering terms a_n with $n \leq 150$ and $a_n \leq 10^{150}$).

For 6719 sequences the terms are consistent with being Stieltjes

$6719 - \epsilon$ **open questions:** Which of these sequences are really Stieltjes?

Refined results:

- In 1667 such cases, one of the terms $\alpha_j = 0$, so the generating function $A(t)$ is rational
- In 798 cases (including 328 rational cases), the coefficients α_j are all integers.
- For 7344 sequences the first 15 terms are consistent with being Stieltjes (625 of these not Stieltjes because of later terms)

S-fraction applications

Using continued fractions identities:
Many Multivariate polynomial generalisations of:
Catalan numbers, $n!, (2n-1)!!$, Genocchi numbers, $n^{n-1,2,3}$, tangent and secant numbers, Jacobi elliptic functions, ...

(*many papers by Euler,..., Gauss, ..., Stieltjes, Rogers, Ramanujan, ..., Flajolet, Viennot, Zeng,..., Corteel, Sokal, D., ...*)

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- Randrianarivony, European J. Combin., 1998.

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$$\sum_{n=0}^{\infty} T_n(\boldsymbol{\alpha}, \boldsymbol{\delta}) t^n := \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \delta_1 t - \cfrac{\alpha_2 t}{1 - \ddots}}}.$$

Then $(T_n(\boldsymbol{\alpha}, \boldsymbol{\delta}))_{n \geq 0}$ *is coefficientwise Hankel-TP.*

T-fractions applications

- Elvey Price–Sokal, Electronic J. Combinatorics, 2020.

  Super-augmented perfect matchings: the most general combinatorial model for T-fractions with linear α, δ
  Generalisations of Ward numbers, counting phylogenetic trees, etc.

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  D-permutations: multivariate continued fractions counting many statistics on D-permutations

- Bitonti–D.–Sokal, Electronic J. Combin., 2026.

  Increasing interval-labeled restricted ternary trees: combinatorial model for T-fractions with quasi-affine $\boldsymbol{\alpha}, \boldsymbol{\delta}$.
  Performed exhaustive search of OEIS for sequences with quasi-affine T-fraction coefficients of period 2.
  (Talk by Veronica Bitonti on Wednesday)

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Define matrix $A = (a_{n,k})_{n,k \geq 0}$ where $a_{n,k} = (P^n)_{0k}$

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Existence of S-fraction is a special case.
When P is tridiagonal matrix $a_{n,0}$ counts Motzkin paths of length n: corresponds to Jacobi-type continued fractions

Jacobi matrix from Orthogonal polynomial theory

Production matrix method for proving coefficientwise Hankel total positivity

Consists of two steps:

1. Guess production matrix and prove that it is the production matrix.
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The hardest part is usually to guess the production matrix.

Applications of the production matrix method

- "Lattice paths and branched continued fractions I"
  Pétréolle–Sokal–Zhu, Memoirs A.M.S., 2023.
  multivariate generalisations involving various counts of lattice paths, higher-order Stirling permutations, ratios of hypergeometric series, etc.

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Enumeration of Trees and forests

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  Pétréolle–Sokal, European J. Combin. 2021:
  Multivariate Lah polynomials
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  Enumeration of trees and functional digraphs

Also see Gilmore, Electron. J. Combin. (2021) for LGV-digraph approach.

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Multivariate Laguerre polynomials

- Zhu, Adv. Math., 2024.
- D.–Dyachenko–Pétréolle–Sokal (arxiv 2023)

Few recent papers

- D.–Sokal, to appear in SIAM J. Discrete Math., 2026?.
- Zhu, JLMS, 2026.
- Ding–Mu–Zhu, Adv. App. Math., 2026.

Branched Stieltjes-type continued fractions

The special case when production matrix has P the following decomposition:

$$P = L_1 L_2 \cdots L_m U$$

where the L_i are lower bidiagonal matrices and U is upper bidiagonal matrix.
These are branched Stieltjes-type continued fractions. When $P = LU$: S-fractions.
In this case, P is totally positive.

In most applications involving the production matrix method, there is often an underlying (exponential) Riordan array. Many of the previously listed papers have this underlying structure.

Exponential Riordan Arrays

$F(t)$ and $G(t)$: formal power series with $G(0) = 0$

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The **exponential Riordan array** $\mathcal{R}[F, G]$ is the infinite lower-triangular matrix with entries

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$$\mathcal{R}[F, G]_{nk} = \frac{n!}{k!} [t^n] F(t) G(t)^k .$$

Equivalently, the bivariate exponential generating function of $\mathcal{R}[F, G]$ is

$$\sum_{n,k=0}^{\infty} \mathcal{R}[F, G]_{nk} \frac{t^n}{n!} x^k = F(t) e^{xG(t)} .$$

A and Z sequences

Let $\mathbf{a} = (a_n)_{n\geq 0}$ and $\mathbf{z} = (z_n)_{n\geq 0}$ be two sequences with ordinary generating functions $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $Z(t) = \sum_{n=0}^{\infty} z_n t^n$

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Equivalently

$$A(s) = G'(\bar{G}(s))$$

$$Z(s) = \frac{F'(\bar{G}(s))}{F(\bar{G}(s))}$$

Production matrix for exponential Riordan arrays

Theorem (See (Barry, '16))

The production matrix P of an exponential Riordan array $\mathcal{R}$ with A sequence $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ is given by

$$P_{nk} \;=\; \frac{n!}{k!}(z_{n-k} + k\, a_{n-k+1})$$

Alternate proof in
D.–Sokal, Jacobian elliptic and hyperelliptic functions, Schett polynomials and Schett symmetric functions, and coefficientwise Hankel-total positivity , in preparation.

If the sequence can be extended to fill an exponential Riordan array, we have a nice formula for the production matrix.
We then hope that the production matrix is TP.

Applications: Production matrix method for exponential Riordan arrays

After existence of S-fractions, most powerful tool in our arsenal

- Pétréolle–Sokal, European J. Combin. 2021.
- Sokal, Monatshefte Math., 2022.
- Chen–Sokal, Adv. App. Math., 2024.
- Zhu, Adv. Math., 2024.
- D.–Dyachenko–Pétréolle–Sokal (arxiv 2023)
- Bitonti–D.–Sokal, Electronic J. Combin., 2026.
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General theory of exponential Riordan arrays with a checkerboard shape

- D.–Sokal, Jacobian elliptic and hyperelliptic functions, Schett polynomials and Schett symmetric functions, and coefficientwise Hankel-total positivity , in preparation.

Sokal, Trans. A.M.S. 2024.

Theorem

The output matrix of a production matrix P corresponds to the signed inverse of the coefficient matrix of a sequence of multiple orthogonal polynomials along the stepline recurrence.

Toeplitz TP implying Hankel TP: two results

Let R be a partially ordered commutative ring. Let $(\phi_n)_{n\geq 0}, (b_n)_{n\geq 0}$, be two sequences in R. Let F be some transformation function on sequences in R such that $F((\phi_n)_{n\geq 0}) = (b_n)_{n\geq 0}$.

Several theorems in the subject have the following form:

Type of theorem statements: If the sequence $(\phi_n)_{n\geq 0}$ is Toeplitz totally positive then the transformed sequence $(b_n)_{n\geq 0}$ is Hankel totally positive.

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We now state two such theorems.

Theorem 1: Multivariate Ward, Zhu, Adv. Math. 2024

The multivariate Ward polynomials were introduced by Elvey Price–Sokal (2020).
Let $W_n(\phi_1, \phi_2, \ldots)$ be the generating polynomial for phylogenetic trees (leaf-labeled rooted tree in which all non-leaf vertices have ≥ 2 children) on $n+1$ labeled leaves in which each vertex with $i \geq 2$ children gets weight ϕ_{i-1}.

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Theorem (Zhu, 2024)

Let the Taylor coefficients of the power series

$$\frac{1}{1 - \sum_{i \geq 1} \phi_i \frac{t^n}{n!}}$$

be Toeplitz totally positive.
Then the sequence of multivariate Ward polynomials $(W_n(\phi_1, \phi_2, \ldots))_{n \geq 0}$ *is Hankel totally positive.*

Theorem 2: Multivariate Lah, Pétréolle–Sokal, European J. Combin. 2021

Let $T_{n,k}$ be set of all unordered forests of increasing ordered trees with vertices $\{1, \ldots, n\}$ with k connected components. We assign weight ϕ_i to each vertex with i children.
Let $L_{n,k}(\phi)$ be the sum of all weights of trees in $T_{n,k}$.

Theorem (Pétréolle–Sokal, 2021)

Let $(\phi_i)_{n\geq 0}$ be Toeplitz totally positive. Then

(a) *The matrix $(L_{n,k}(\phi))_{n,k\geq 0}$ is totally positive.*

(b) *The sequence $(\sum_{k=0}^{n} L_{n,k}(\phi) y^k)_{n\geq 0}$ is Hankel totally positive.*

(c) *The sequence $(L_{n+1,1}(\phi))_{n\geq 0}$ is Hankel totally positive.*

Only sufficient but far from necessary

For the previous theorem, $(\phi_n)_{n\geq 0}$ being Toeplitz TP is only a sufficient condition but far from necessary.

- D.–Sokal, SIAM Discrete Math. (2026?):
  Define $\phi_0 = 1$, $\phi_i = x(1+x)^{i-1}$. Then the sequence is $(L_{n+1,1})_{n\geq 0}$ is coefficientwise Hankel totally positive even though (ϕ_i) is not a Toeplitz TP sequence.

- D.–Dyachenko–Sokal, in preparation.
  We take $\phi_i = 0$ for all $i \geq 4$ and $\phi_0, \phi_1, \phi_2, \phi_3 \in \mathbb{R}$. We did explicit experiments to compute the exact region of necessary and sufficient conditions under which $(L_{n+1,1})_{n\geq 0}$ is coefficientwise Hankel totally positive.

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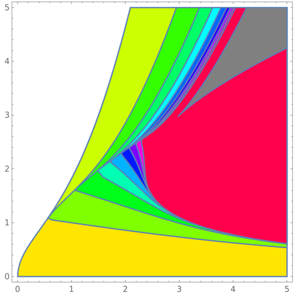

Grey region is the region from Petreolle–Sokal (2021), the red region is converging to the true necessary and sufficient region.

Triangular convolution preserving Stieltjes moment property

Theorem (Wang–Zhu, Adv. App. Math. 2016)

Let $A = (a_{n,k})_{n,k\geq 0}$ be a lower triangular matrix, and let (b_n) and (c_n) be Stieltjes moment sequences. If the sequence of row-generating polynomials of the matrix A is coefficientwise Hankel totally positive then the sequence (z_n) where

$$z_n = \sum a_{nk} b_k c_{n-k}$$

is also a Stieltjes moment sequence.

Also works if condition on row-generating polynomials is true pointwise.

Operators preserving CHTP

Problem: Which operators preserve coefficientwise Hankel total positivity?

Things to work out in great detail

- Connection to integrability
- Connection to free probability
- Scaling limits of combinatorial objects

A new web repository of problems

https://bishaldeb.com/positivity.html

(Lake district hike organised by Natasha Blitvic and Tomack Gilmore, 2022)

Happy 71st birthday Alan